

Learning & Teaching

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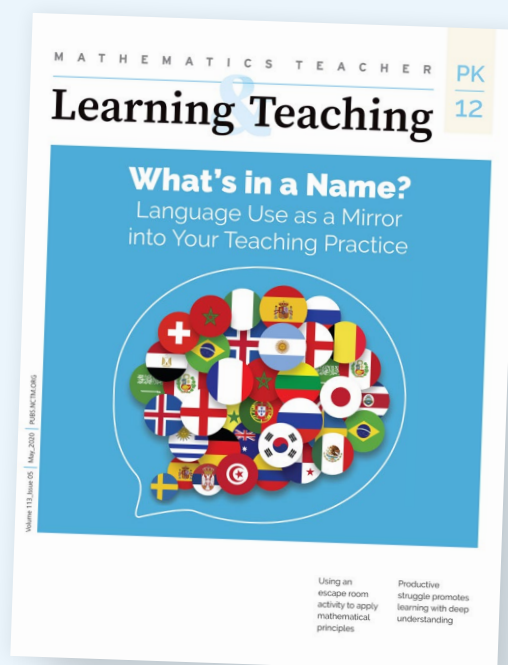
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Adaptive Teaching: Learning Trajectories

Adaptive teachers use student data to guide instruction. Learn about
using an anecdotal record form to support adaptive teaching.

Melissa A. Gallagher and Jennifer E. Scholla

Each day, as students walk into their classrooms, they bring with them vast and varied past experiences. These experiences include rich cultural traditions, challenging home events, and various interactions with mathematical concepts. When students enter the classroom of an adaptive teacher, that teacher takes the time to learn about each student's social, emotional, and academic needs and adapts their instruction to meet those needs (Vaughn et al., 2022). In this article, we describe how teachers can use learning trajectories to make adaptive decisions to meet the needs of their students. We provide an

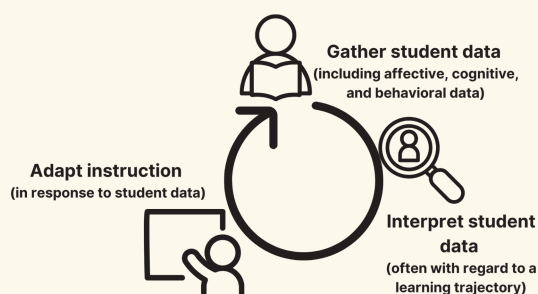
example using the U.S. Math Recovery Council's (link online) learning trajectories and sub-goals for addition and subtraction to 100. We offer concrete tools and suggestions for how teachers can elicit, record, and interpret student thinking using learning trajectories to adapt their instruction to meet the needs of the students in their classes.

ADAPTIVE TEACHING

Adaptive teaching can be defined as the thoughtful modifications teachers make to instruction to meet

their students' specific needs. It can describe decisions made in the moment of teaching or decisions made after or before teaching when planning for next steps. In Figure 1, we share the Adaptive Teaching Feedback Loop, which is a summary of decades of research on adaptive teaching in mathematics (Gallagher et al., 2022). This feedback loop describes the process that many teachers go through, often without even realizing it: (a) They observe some information or data that comes from their students; (b) they interpret this observation in relation to what they know about teaching and learning and decide how to respond; and (c) they take action—no action can be considered an action as well—to respond to the initial observation. Teachers' interpretations and decisions to act can be influenced by many different factors, affordances, and barriers, such as their knowledge, beliefs, and experience or the curriculum they use. Scripted curricula, in particular, can be a major barrier to adaptive teaching (Vaughn et al., 2022). Although these actions in and of themselves may be described as teaching, we argue that adaptive teachers do not just act, but rather thoughtfully consider their students' varied needs and backgrounds to choose how to act in a way that helps move their students' learning forward.

Figure 1 Adaptive Teaching Feedback Loop



Note. Adapted from Gallagher et al. (2022).

Student data can come in many different forms, including observational data on how engaged they are in an activity, but the most useful data for adapting mathematics instruction are data on students' mathematical conceptions and strategies that teachers can observe either on assessments or during instruction. By systematically recording and interpreting these data, teachers can determine the next best instructional steps for their students, but this is not an easy task.

Eliciting, Recording, and Interpreting Students' Data

Eliciting, recording, and interpreting students' thinking are core practices used by adaptive teachers (Franke et al., 2009; National Council of Teachers of Mathematics, 2014). Eliciting involves the "teacher draw[ing] out a student's thinking through carefully chosen questions and tasks" (TeachingWorks, n.d.). Key to the practice of eliciting is using open-ended questions to gather as much data as possible, which the teacher can then use to interpret what mathematical conceptions the student holds (i.e., what they know) and what strategies they are comfortable using (i.e., what they can do). Although many teachers use rich tasks and open-ended questions to begin to elicit student thinking, it can be challenging to keep track of which student exhibits each conception and strategy. For this reason, we encourage teachers to not just elicit and interpret, but rather elicit, *record*, and interpret their observations of their students' conceptions and strategies.

Anecdotal records are one tool that teachers can use to record students' conceptions and strategies as they emerge during assessment, whole or small group instruction, group discussions, one-on-one with the teacher, or in written work. There are many ways teachers can organize these records (Bates et al., 2019). In our work, we have found that organizing these around established learning trajectories can be helpful both for capturing students' conceptions and strategies

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in the moment and as a frame of reference for asking eliciting questions. Specifically, as a teacher begins to record which students are demonstrating specific conceptions and strategies, they will also notice conceptions and strategies that have not yet emerged in the discussion and can ask questions with the specific intent of eliciting those conceptions and strategies. Recording this data for each student over time can help teachers understand and track each student's progress.

Learning Trajectories

Learning trajectories, which are grounded in decades of research on student thinking, provide a comprehensive list of conceptions and strategies to look for and can be a useful tool for adapting teaching based on students' needs (Confrey et al., 2014; Sarama & Clements, 2019). The U.S. Math Recovery Council supports teachers in learning how to elicit, record, and interpret students' thinking using tools aligned to the Learning Framework in Number (LFIN), a compilation of early number learning trajectories (Wright & Ellemor-Collins, 2018), including the following domains:

- Number Words & Numerals
- Early Structuring & Structuring Numbers 1–20
- Early Arithmetical Strategies
- Conceptual Place Value
- Addition & Subtraction to 100
- Early Multiplication and Division
- Multiplicative Strategies and Basic Facts

Knowledge and skills developed in one domain support and connect to knowledge and skills in another. These trajectories provide a lens through which teachers can observe students and hypothesize about what they know and understand and what might be the next best instructional steps (i.e., to thoughtfully adapt their instruction). The mathematical domains within the LFIN build and support a robust mathematical framework for students.

ADAPTIVE TEACHING FOR ADDITION AND SUBTRACTION TO 100

Let's examine the domain of Addition and Subtraction to 100. Adding and subtracting two- and three-digit numbers builds extensively off addition and subtraction to 20 and conceptual place value (the ability to flexibly increase and decrease by ones, tens, and hundreds) and works to develop flexible mental computation

for addition and subtraction in the range of 1–100. Skills, knowledge, and understanding in working with numbers 1–100 can lead to more formal mathematics beyond 100 (e.g., semi-formal and formal algorithms for addition and subtraction). Here, we present some of the key mathematical conceptions and strategies that comprise this learning trajectory (see Table 1), an anecdotal record form for making note of students' thinking on the learning trajectory, examples of how students may demonstrate these conceptions and strategies (see Table 2), and possible pedagogical moves teachers can use to thoughtfully adapt instruction to help their students progress through the trajectory.

Learning Trajectory for Addition and Subtraction to 100

Each mathematical domain in the LFIN is made up of learning trajectories, which are broken down into sub-goals that represent a progression of student understanding and strategy use. The sub-goals below are part of the Addition & Subtraction to 100: Mental Strategies, a Ready Set Math instructional supplement provided by the U.S. Math Recovery Council grounded in the LFIN trajectories (U.S. Math Recovery Council, n.d.). While the sub-goals describe the kinds of tasks students should be able to solve as they move through the learning trajectory, task type alone is not the trajectory. It is the nature of a student's solution strategy, rather than the kinds of tasks they can solve, that indicates the knowledge and skills available to them. Considering both the task as well as the student's strategy is crucial when eliciting, recording, and interpreting students' thinking. Interpreting students' thinking in relation to a learning trajectory can help to identify what students already know and help to determine the next instructional steps.

Anecdotal Records for Addition and Subtraction to 100

Initial adaptive decisions take place before instruction to determine where individual students and groups of students are on a learning trajectory. Using an anecdotal record form that connects to the learning trajectory can help teachers interpret where their students are along the trajectory and highlight the strengths their students bring (see Figures 2 and 3). Teachers use this knowledge to make informed decisions regarding which instructional tasks to select, materials to use to scaffold student learning, or teacher moves to support conceptual learning. Next, we share

Table 1 Sub-Goals and Knowledge Checks for Addition and Subtraction to 100

Sub-Goal	Description	Student Knowledge Checks
Sub-Goal A: Use non-count-by-one strategies to add or subtract a one-digit number to or from a two-digit number	Non-count-by-one strategies for adding or subtracting a one-digit number to/from a two-digit number involve applying knowledge of number relationships used for addition and subtraction in the range of 20 to higher decades. Tasks might stay within a decade family ($76 + 3$ or $56 - 4$) or might require crossing a decuple ($68 + 5$ or $42 - 7$).	a. Ask students to mentally solve addition and subtraction tasks involving a two-digit number $\pm$ a one-digit number that stays within a decade family (e.g., $43 + 6$ and $78 - 4$). b. Ask students to mentally solve addition and subtraction tasks involving a two-digit number $\pm$ a one-digit number that crosses to a new decade family (e.g., $67 + 5$ and $82 - 6$). If students are not successful or need to count on or back by ones, check first to see if they can solve addition and subtraction in the range to 20 without counting.
Sub-Goal B: Use initial mental strategies for two-digit addition and subtraction	Initial mental strategies include two main approaches for operating with tens and ones: Jump and Split. The Jump strategy involves starting at one number and then counting by 10s and by 1s (solving $46 + 21$ as $46 \dots 56, 66, 67$). The Split strategy involves combining units of the same rank (solving $64 + 23$ as $60 + 20$ and $4 + 3$; $80 + 7$).	a. Ask students to mentally solve two-digit addition and subtraction tasks that do not require regrouping (e.g., $46 + 31$ and $65 - 23$). Use careful observation and questioning to determine the use of Jump or Split approaches. If students are inaccurate, need to count on or back by ones, or are limited to only mimicking standard written procedures, check first for the prerequisite knowledge of addition and subtraction to 20.
Sub-Goal C: Flexibly use mental strategies for two-digit addition and subtraction	Flexible use of mental strategies involves developing and applying a range of strategies that leverage number relationships. This includes such things as using “friendly numbers” and recognizing useful combinations and partitions. Students who use both Jump and Split strategies, and variations of these are demonstrating strong base-10 knowledge.	a. Ask students to mentally solve a range of two-digit addition and subtraction tasks, including those that require regrouping, which, in the case of mental strategies, is a reorganization of the tens and ones. (e.g., $68 + 24$, $37 + 19$, $93 - 28$, and $41 - 23$). Look for flexible thinking and the use of a variety of strategies across multiple tasks. If students are not successful, need to count on or back by ones, or are limited to only mimicking standard written procedures, check first for knowledge related to Sub-goal B.
Sub-Goal D: Refine and extend use of mental computation strategies	Refining and extending the use of mental strategies involves relying on relational thinking and drawing on number relationships. This includes exploring, comparing, and evaluating strategies. Tasks advance to adding multiple two-digit numbers and to adding or subtracting reasonable three-digit numbers.	Mental computation through 100 provides a basis for the development of meaningful computation methods beyond 100. While the learning trajectory places significance on the development of mental strategies in the range of 100, extending the range to reasonable three-digit tasks is one way to increase the complexity of learning by applying number sense and relational thinking in contexts beyond 100 and to lay the groundwork for semi-formal and formal computation strategies. a. Ask students to mentally solve addition tasks involving three, two-digit numbers (e.g., $32 + 45 + 21$, $28 + 27 + 34$, and $86 + 21 + 52$) b. Ask students to describe what strategy or strategies would work well for certain tasks, looking at how they make strategic choices (e.g., $167 + 29$ might elicit an over-jump compensation [$167 + 30 - 1$]; $482 - 23$ might elicit a ‘jump to the decuple’ strategy [$- 2$, then $- 20$, then $- 1$]). c. Ask students to mentally solve addition and subtraction tasks involving reasonable three-digit numbers (e.g., $780 + 43$, $638 + 120$, $436 - 37$, $827 - 209$). If students are unsuccessful, only mimic standard written procedures, and/or do not demonstrate insight with strategy selection and use, check for knowledge related to Sub-goal C.

Note. Adapted from *Ready, Set, Math: Resources to Enhance Numeracy in the Classroom* by the U.S. Math Recovery Council (<https://www.mathrecovery.org/ready-set-math/modules>).

task selection, implementation, and student solutions for the trajectory Addition and Subtraction within 100.

Student Thinking With Addition and Subtraction to 100

We will take as our example a standard task found in a second- or third-grade classroom when working in this domain, such as $58 + 23$. Posing this task in a small,

guided group allows all students space to share their understandings and affords the teacher the opportunity to record this thinking. It is helpful to consider the knowledge needed to solve such a task before observing students' strategies. For example, we know that the task:

- is additive,
- is a "regrouping" task,

Table 2 Student Solutions to $58 + 23$

Student	Observe	Interpret	Next Steps
Olivia	58 plus 20 more. . . . 68. . .78. Then I need to add 3 more. . .78 + 2 is 80. . .plus 1 more is 81 .	• Recognizes 23 is made up of a 20 and a 3; • Can increment by 10s off the decuple (58, 68, 78); • Partitions 3 into 2 and 1 quickly; • Adds across the decuple; • Uses a non-count-by-ones strategy to find the answer accurately.	Olivia is likely working within Sub-Goal B or beyond. To gather more information or extend learning, consider prompting for additional strategies or extending learning by varying the orientation of tasks, including missing addend, subtraction, or missing subtrahend problems.
Noah	58. . .[counting on fingers. . .] 59, 60, 61, 62, 63, [skips 64], 65, 66, 67, 68, 69,so 10 more is 69. . .another 10. . .would that be 79? [pause. . .counts 10 more quietly using fingers] yeah, that's 79. Plus 3 more. . .80, 81, 82 . (counting by 1s strategy)	• Works off of 58 as a composite unit of 58 ones; • Can count on from 58 by ones, although with an error; • Recognizes 23 is made up of two tens and a three; • Recognizes "10 more," but confirms with counting by 1s – not yet incrementing by 10s.	Both Noah and Emma would likely be working in Sub-Goal A. For both students, consider providing additional scaffolding, such as sticks bundled into tens and ones, which support incrementing/ decrementing by 10s and 1s, provide opportunities for self-checking and reflection, and support identifying the reasonableness of one's response. It may also be beneficial to assess pre-requisite knowledge (e.g., a deep understanding of 10s and 1s or addition and subtraction to 20) to determine if any related skills might support their overall understanding.
Emma	$5 + 2$ is 7, and $8 + 3$ is 11. . .so. . . 711 ?	• Knows the facts $5 + 2$ and $8 + 11$ by memory; • Is not yet considering the value of the digits in her computation strategy nor considering the reasonableness of her answer.	
Liam	Well. . .it's easier to do $60 + 23$ first. . .because I'm just making it a 10, so that is 83. . .but 60 was 2 more than 58, so I have to go back 2. . .which leaves me with 81 . See how that works? It's just like doing $58 + 20$ more is 78, plus 3 more is 81. . .but you're adding 2 on in the beginning, so you have to take 2 off in the end.	• Uses the relationships between the numbers to choose an efficient strategy; • Tracks own thinking; • Manipulates numbers, focuses on different strategies rather than a single strategy and evaluates strategies in the process of solving; • Solves in more than one relatively sophisticated and efficient way.	Liam is likely working in Sub-Goal C or beyond and is likely ready to extend strategies to add and subtract beyond 100, if he is not already doing so.

- requires some understanding of “tens” and “ones,”
- likely requires more efficient strategies (beyond counting from 1), and
- can be solved using a variety of strategies, including the standard algorithm.

Because the task is two-digit plus two-digit, it is likely more accessible to students working within Sub-Goals B-D. Therefore, teachers may plan before instruction for ways to create access for students who have primarily demonstrated strategies related to sub-goal A.

Observing how students solve tasks and eliciting their understanding can help teachers develop hypotheses about where they are in their learning trajectory. Although a single task should not be used to place a student on the trajectory (we consider strategies over a range of tasks), it can help teachers consider how to be adaptive. Table 2 includes a description of the strategies used by four students in our classrooms along with our interpretation of these using the learning trajectory sub-goals for Addition and Subtraction to 100. We recorded these students’

Figure 2 Anecdotal Record Form for Adding and Subtracting to 100 (Front Side)

ADDING & SUBTRACTING TO 100 GROUP RECORDING SHEET				
Student Names	Sub-Goal A		Sub-Goal B	
	Use non-count-by-one strategies to add or subtract a 1-digit number to or from a 2-digit number. • Add and subtract to and from a decade • Use partners to 10 (e.g., 7 and 3) • Partition 1-digit numbers (e.g., 4 is 2&2)		Use initial mental strategies for 2-digit addition and subtraction without regrouping. • Use the Jump strategy (e.g., $16+2 \rightarrow 16, 56, 66, 67$) • Use the Split strategy (e.g., $16+2 \rightarrow 10+6$ and $6+1=7$, so 67)	
Olivia				
Noah	$58+23$ 			
Emma	$58+23$ $58+2=60$ $60+3=63$ $63+10=73$ $73+1=74$			
Liam				

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Figure 3 Anecdotal Record Form for Adding and Subtracting to 100 (Back Side)

ADDING & SUBTRACTING TO 100 GROUP RECORDING SHEET				
Student Names	Sub-Goal C		Sub-Goal D	
	Flexibly use mental strategies for 2-digit addition and subtraction with regrouping. • Use friendly numbers (e.g., partners to 10) • Use the Jump strategy • Use the Split strategy • Other strategies that demonstrate strong base-10 knowledge		Refine and extend use of mental computation strategies. • Apply strategies from sub-goal C to adding and subtracting more than two 2-digit numbers or adding and subtracting 3-digit numbers • Evaluate and select strategies based on relational thinking and number relationships	
Olivia				
Noah				
Emma				
Liam	$58+23$ 			

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thinking in the anecdotal record form (see Figures 2 and 3). All names are pseudonyms.

Although teachers might quickly recognize the incorrect answers in both Noah's and Emma's thinking, looking beyond the answers and discovering what the student knows and is able to do can help determine current understanding and help to guide the next best instructional steps. We have highlighted four students' strategies, but we recognize that these are just a sampling of the wide range of strategies educators see in classrooms. By attending to how students are solving and considering the trajectory of learning, teachers can thoughtfully adapt the next instructional steps for students.

IMPLEMENTING IN A CLASSROOM SETTING

For a teacher to have the opportunity to use anecdotal records in this way, a few aspects of the instructional environment need to be established: (a) rich student talk (e.g., including number talks, students solving rich tasks, playing mathematics games) and (b) small and whole group instruction. We encourage the use of the mathematics workshop model (Lempp, 2017), where teachers begin each lesson with a mathematics routine; meet with small, guided groups regularly; and engage students in mathematics games and rich tasks. Mathematics workshops create opportunities for a teacher to take anecdotal notes for two to three children each day, such that over a two-week period, the teacher can make a note about each student. A teacher might implement the adaptive teaching feedback loop

(Figure 1) by making a note of a child's thinking if they appear to be at a different point on the learning trajectory than her instruction (i.e., gathers student data). For example, if a teacher's goal is for students to be working in Sub-Goal C, but they notice a student not using partners to 10 (i.e., interprets student data), this would be a good observation to record. The teacher might ask that student to play a partners to 10 game during the mathematics workshop the subsequent day (i.e., adapt instruction in response to student data). Many teachers make these kinds of observations informally, but recording these notes can help ensure that they are systematically collected. A recording sheet organized around a learning trajectory can help the teacher pinpoint exactly what students need.

HONING YOUR ADAPTIVE TEACHING SKILLS

Adaptive teaching is a hallmark of expert teachers (Corno, 2008; Parsons et al., 2018), but it is not an easy skill to develop. We have provided concrete examples of a learning trajectory for adding and subtracting to 100, a recording sheet, and how these might be used to elicit, record, and interpret the thinking of four second-grade students in the supplementary materials ([link online](#)). We encourage you to either (a) try out the task we have shared in a small mathematics group and record the thinking of two of your students on the anecdotal record form or (b) find a learning trajectory and task related to the content you are currently teaching (see activities [[link online](#)] and learning trajectories [[link online](#)], or Wright & Ellemor-Collins, 2018). [—](#)

REFERENCES

- Bates, C. C., Schenck, S. M., & Hoover, H. J. (2019). Anecdotal records. *Young Children*, 74(3), 14–19.
- Confrey, J., Maloney, A. P., & Corley, A. K. (2014). Learning trajectories: A framework for connecting standards with curriculum. *ZDM Mathematics Education*, 46, 719–733. <https://doi.org/10.1007/s11858-014-0598-7>
- Corno, L. Y. N. (2008). On teaching adaptively. *Educational Psychologist*, 43(3), 161–173. <https://doi.org/10.1080/00461520802178466>
- Franke, M. L., Webb, N. M., Chan, A. G., Ing, M., Freund, D., & Battey, D. (2009). Teacher questioning to elicit students' mathematical thinking in elementary school classrooms. *Journal of Teacher Education*, 60(4), 380–392. <https://doi.org/10.1177/0022487109339906>
- Gallagher, M. A., Parsons, S. A., & Vaughn, M. (2022). Adaptive teaching in mathematics: A review of the literature. *Educational Review*, 74(2), 298–320. <https://doi.org/10.1080/00131911.2020.1722065>
- Lempp, J. (2017). *Math workshop: Five steps to implementing guided math, learning stations, reflection, and more, grades K-6*. Math Solutions.
- National Council of Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematics success for all*.
- Parsons, S. A., Vaughn, M., Scales, R. Q., Gallagher, M. A., Davis, S. G., Parsons, A. W., Pierczynski, M., & Allen, M. (2018). Teachers' instructional adaptations: A research synthesis. *Review of Educational Research*, 88, 205–242. <https://doi.org/10.3102/0034654317743198>
- Sarama, J., & Clements, D. H. (2019). Learning trajectories in early mathematics education. In D. Siemon, T. Barkatsas, & R. Seah (Eds.), *Researching and using progressions (trajectories) in mathematics education* (pp. 32–55). Brill Sense.
- TeachingWorks. (n.d.). *High leverage practices*. <http://www.teachingworks.org/high-leverage-practices>
- U.S. Math Recovery Council. (n.d.). *Ready, set, math: Resources to enhance numeracy in the classroom*. <https://www.mathrecovery.org/ready-set-math/modules>
- Vaughn, M., Parsons, S. A., & Gallagher, M. A. (2022). Challenging scripted curricula with adaptive teaching. *Educational Researcher*, 51(3), 186–196. <https://doi.org/10.3102/0013189X211065752>
- Wright, R. J., & Ellemor-Collins, D. (2018). *The Learning Framework in number: Pedagogical tools for assessment and instruction*. Sage.

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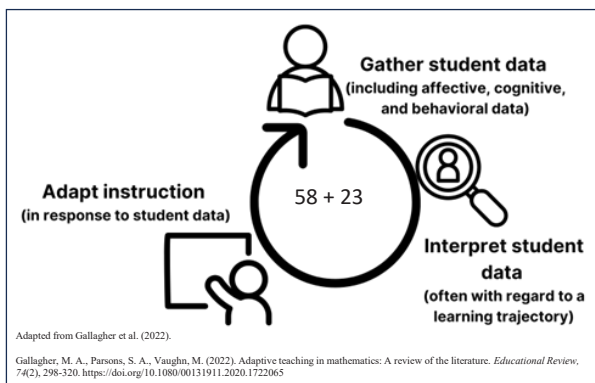
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Adaptive teaching can be defined as the thoughtful modifications teachers make to instruction to meet their students' specific needs. It can describe decisions made in-the-moment of teaching, or decisions made after or before teaching when planning for next steps.

Possible Adaptations:

- Prompt for additional strategies
- Plan for additional tasks, considering various task orientations such as missing addend, subtraction, or missing subtrahend.
- Consider extending the range into 3-digit addition or subtraction.



Gather data:

Observe student thinking aloud "58 plus 20 more...58...78. [Pause] Now I need to add 3 more...78 + 2 is 80, so plus 1 more is 81."

Interpret data:

Use learning trajectory as lens for recognizing what student knows:

- Recognizes 23 is made up of 20 and 3
- Can increment by 10s off the decuple (58, 68, 78)
- Partitions 3 into 2 and 1 quickly
- Adds across the decuple
- Uses a non-count-by-1 strategy to find the answer accurately.



Learning trajectories, such as those provided with the Ready Set Math instructional supplemented by US Math Recovery Council, can provide a lens through which teachers can observe students and hypothesize about what they know and understand and what might be the best instructional steps (i.e., to thoughtfully adapt their instruction).